The critical equation of state of the two-dimensional Ising model

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Abstract

We compute the $2n$-point coupling constants in the high-temperature phase of the two-dimensional Ising model by using transfer-matrix techniques. This provides the first few terms of the expansion of the effective potential (Helmholtz free energy) and of the equation of state in terms of the renormalized magnetization. By means of a suitable parametric representation, we determine an analytic extension of these expansions, providing the equation of state in the whole critical region in the t, h plane.

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1. Introduction

Despite its apparent simplicity and the fact that for more than 50 years the exact expression of the free energy along the $h = 0$ axis has been exactly known, the two-dimensional Ising model still provides many interesting issues. Since the original Onsager solution [1], several other exact results have been obtained for this model. In particular, closed expressions for the two-point correlation function at $h = 0$ and an S-matrix solution on the $t = 0, h \neq 0$ axis [2] exist (for a review, see [3]). However, little is known for generic values of t and h and, in particular, there is no exact expression for the free energy and for the critical equation of state in the whole (t, h) plane.

In order to determine the Helmholtz free energy (also called the effective potential) and the equation of state, we begin by computing the first terms of their expansion in powers of the magnetization in the high-temperature (HT) phase. The coefficients of this expansion are directly related to the n-point zero-momentum renormalized couplings g_n that are also interesting in themselves, since they summarize relevant (zero-momentum) physical properties

of the quantum field theory that describes the Ising model in the vicinity of the critical point. For this purpose we combine transfer matrix (TM) techniques, conformal field theory (CFT) methods and general renormalization-group (RG) properties of critical systems. Recent works [4, 5] have shown that a combination of these approaches can lead to very accurate estimates of universal ratios in the two-dimensional Ising model. In this respect this paper is the natural continuation of [5], in which the first nontrivial coupling g_4 was determined with the same techniques as used here.

Starting from the expansion of the free energy in powers of the magnetization in the HT phase, we determine approximate representations of the equation of state that are valid in the critical regime in the whole (t, h) plane. This requires an analytic continuation in the complex t plane [6, 7], extending the expansion valid for $t > 0$ to the low-temperature phase $t < 0$. For this purpose, we use parametric representations [8–10], which implement in a rather simple way the known analytic properties of the equation of state (Griffiths' analyticity). We construct a systematic approximation scheme based on polynomial parametric representations [7] and on a global stationarity condition [11]. This approach was successfully applied to the three-dimensional Ising model, leading to an accurate determination of the critical equation of state and of the universal amplitude ratios that can be obtained from it [7, 11].

The paper is organized as follows: in section 2 we set up the formalism and define the quantities that parametrize the free energy for small values of the magnetization. These coefficients are evaluated numerically in section 3 by using TM techniques. In section 4 we use a parametric representation to analytically extend this expansion to the whole critical region, obtaining approximate, but still quite precise, expressions of the critical equation of state in the whole (t, h) plane. Finally in section 5 we draw our conclusions. In appendix A we report our notations for the universal amplitude ratios, in appendix B we present a detailed discussion of the finite-size behaviour of the free energy and of its derivatives and in appendix C we explicitly compute e_h, one of the coefficients appearing in the expansion of the nonlinear scaling field associated with the magnetic field.

2. Small-field expansion of the effective potential in the high-temperature phase

In the theory of critical phenomena continuous phase transitions can be classified into universality classes determined only by a few basic properties characterizing the system, such as space dimensionality, range of interaction, number of components and symmetry of the order parameter. RG theory predicts that, within a given universality class, the critical exponents and the scaling functions are the same for all systems. Here we consider the two-dimensional Ising universality class, which is characterized by a real order parameter and effective short-range interactions. A representative of this universality class is the standard square-lattice Ising model defined by the partition function

$$Z = \sum_{\sigma_i = \pm 1} e^{\beta \sum_{\langle n,m \rangle} \sigma_n \sigma_m + h \sum_n \sigma_n} \tag{2.1}$$

where the field variable σ_n takes the values $\{\pm 1\}$, $n \equiv (n_0, n_1)$ labels the sites of a square lattice of size $L_0 \times L_1$ and $\langle n, m \rangle$ denotes a lattice link connecting two nearest-neighbour sites. In our calculations we treat asymmetrically the two directions. We denote by n_0 the 'time' coordinate and by n_1 the 'space' one. We indicate by $N \equiv L_0 L_1$ the number of sites of the lattice and define the reduced temperature

$$t \equiv \frac{\beta_c - \beta}{\beta_c} \tag{2.2}$$

where $\beta_c = \log(\sqrt{2} + 1)/2 \approx 0.440\,6868\ldots$ is the critical point.

We introduce the Gibbs free-energy density

$$F(t, h) \equiv \frac{1}{N} \log(Z(t, h)) \tag{2.3}$$

and the related Helmholtz free-energy density (in the field-theoretical framework usually called the effective potential)

$$\mathcal{F}(t, M) = M h - F(t, h) \tag{2.4}$$

where M is the magnetization per site. For $t > 0$, $F(t, h)$ and $\mathcal{F}(t, M)$ have regular expansions in even powers of h and M respectively. Explicitly

$$F(t, h) - F(t, 0) = \sum_{n=1} \frac{1}{(2n)!} \chi_{2n}(t) h^{2n} \tag{2.5}$$

$$\mathcal{F}(t, M) - \mathcal{F}(t, 0) = \sum_{n=1} \frac{1}{(2n)!} \chi_{2n}^{1PI}(t) M^{2n} \tag{2.6}$$

where $\chi_{2n}(t)$ is the zero-momentum $2n$-point function and $\chi_{2n}^{1PI}(t)$ is its one-particle irreducible counterpart. Note that $\chi_2(t) = \chi(t)$, where $\chi(t)$ is the magnetic susceptibility.

Expansion (2.6) can also be written in the equivalent forms

$$\mathcal{F}(t, M) - \mathcal{F}(t, 0) = \xi_{2nd}^{-2} \left(\frac{1}{2} \varphi^2 + \sum_{n=2} \frac{1}{(2n)!} g_{2n} \varphi^{2n} \right) \tag{2.7}$$

$$= -\frac{\chi^2}{\chi_4} \left(\frac{1}{2} z^2 + \frac{1}{4!} z^4 + \sum_{n=3} \frac{1}{(2n)!} r_{2n} z^{2n} \right) \tag{2.8}$$

where

$$\varphi^2 = \frac{\xi_{2nd}(t)^2}{\chi(t)} M^2 \qquad z^2 = -\frac{\chi_4(t)}{\chi(t)^3} M^2 \tag{2.9}$$

and ξ_{2nd} is the second-moment correlation length

$$\xi_{2nd} = \frac{1}{4\chi} \sum_x x^2 \langle s(0)s(x) \rangle. \tag{2.10}$$

The coefficients g_{2n} correspond to the $2n$-point renormalized coupling constants and $r_{2n} = g_{2n}/g_4^{n-1}$. The couplings g_{2n} and the related r_{2n} can be computed in terms of χ_{2n} and ξ_{2nd}. The four-point coupling is given by

$$g_4(t) = -\frac{\chi_4}{\chi^2 \xi_{2nd}^2}. \tag{2.11}$$

An explicit expression of r_{2n} in terms of χ_{2n} is given at the beginning of section 3.

The advantage of the expansions (2.7) and (2.8) is the fact that the coefficients have a finite limit—in the following we indicate it by the same symbol—for $t \to 0$. Notice that for $t \to 0$, $z \sim M t^{-\beta}$ and $\chi^2/\chi_4 \sim t^{2\nu}$, so equation (2.8) defines the HT expansion of the scaling part of the free energy $\mathcal{F}(t, M) \sim t^{2\nu} \mathcal{F}_{scal}(M t^{-\beta})$.

3. Calculation of r_{2n} with transfer-matrix techniques

In this section we wish to report the calculation of the first few coefficients r_{2n} at the critical point $t = 0$. It is easy to express them in terms of the critical amplitudes of the $2n$-point functions $\chi_{2n}(t)$. Defining

$$C_n^+ = \lim_{t \to 0} \chi_n(t) t^{15n/8 - 2} \tag{3.1}$$

it is possible to show by direct calculation that

$$r_6 = 10 - \frac{C_6^+ C_2^+}{(C_4^+)^2} \tag{3.2}$$

$$r_8 = 280 - 56 \frac{C_6^+ C_2^+}{(C_4^+)^2} + \frac{C_8^+ (C_2^+)^2}{(C_4^+)^3} \tag{3.3}$$

$$r_{10} = 15\,400 - 4620 \frac{C_6^+ C_2^+}{(C_4^+)^2} + 120 \frac{C_8^+ (C_2^+)^2}{(C_4^+)^3} + 126 \frac{(C_6^+)^2 (C_2^+)^2}{(C_4^+)^4} - \frac{C_{10}^+ (C_2^+)^3}{(C_4^+)^4} \tag{3.4}$$

$$r_{12} = 1401\,400 - 560\,560 \frac{C_6^+ C_2^+}{(C_4^+)^2} + 17\,160 \frac{C_8^+ (C_2^+)^2}{(C_4^+)^3} + 36\,036 \frac{(C_6^+)^2 (C_2^+)^3}{(C_4^+)^4}$$

$$- 220 \frac{C_{10}^+ (C_2^+)^3}{(C_4^+)^4} - 792 \frac{C_8^+ C_6^+ (C_2^+)^3}{(C_4^+)^5} + \frac{C_{12}^+ (C_2^+)^4}{(C_4^+)^5}. \tag{3.5}$$

The aim of this section is to estimate the constants C_{2n}^+, by using TM techniques and the exact knowledge of several terms of the small-t expansion of $\chi_{2n}(t)$. Using the equations reported above, we will then obtain our estimates of r_{2n}.

3.1. The transfer-matrix approach

We obtained our numerical estimates of the $2n$-point functions $\chi_{2n}(t)$ following a two-step procedure. First, by using TM techniques, we obtained estimates of these quantities on lattices of finite transverse size L_1, with $L_1 \leqslant 24$. Second, we extrapolated these results to the thermodynamic limit by using the exact knowledge of their finite-size scaling (FSS in the following) behaviour. To perform this second part of the analysis we have elaborated a new extrapolation scheme, which is rather interesting in itself. We shall discuss it in detail in the last part of this section.

Let us now look at both these steps in more detail.

3.1.1. The TM computation.
This part of the procedure has already been discussed in [5]. We report here the main features of the algorithm for completeness and refer to [5] for a more detailed discussion.

The main idea is to use TM techniques to extract the h-dependence of the magnetization at fixed t. To this end, we computed numerically the magnetization $M(t, h, L_1)$ on lattices of size $\infty \times L_1$, for $L_1 \leqslant 24$. In this paper we use again the data obtained in [5]. In addition, we generated new data for $\beta = 0.335$ and 0.345 and added $L_1 = 24$ results for $\beta = 0.35$ and 0.355. All computations were performed with double-precision floating-point arithmetic on commercial workstations. Typically we obtained M with 15 correct digits.

Then, for each given value of L_1 and β, we determined the coefficients of the series

$$h = b_1 M + b_3 M^3 + b_5 M^5 + \cdots \tag{3.6}$$

which we truncated at order M^{15}. Using the numerical results of M for eight values of h, we computed the coefficients $b_1, b_3, \ldots, b_{15}$. The optimal choice for these eight values of h is the one for which the errors due to the truncation of the series and those due to numerical rounding are of the same magnitude. This optimal range changes as a function of β and L_1. For instance, just to give an idea of the magnitude of the magnetic fields which we studied, for $\beta = 0.36$ and $L_1 = 24$ the best choice is $h = 0.002\,j$ with $j = 1, \ldots, 8$. The accuracy of the b_i obtained in this way is decreasing with increasing order. For example, for $\beta = 0.37$, we obtain b_1 with 14 significant digits and, for example, b_{11} with only three significant digits.

In principle we could have also used the expansion

$$M = a_1 h + a_3 h^3 + a_5 h^5 + \cdots. \tag{3.7}$$

However, in practice, the use of (3.7) leads to results with significantly larger truncation errors, since, in contrast to what happens for b_i, the sign of the coefficients a_i alternates.

Then, for each β and L_1, we constructed the χ_{2n} functions out of the b_i constants. These are the inputs of the FSS analysis discussed in the next section. The relations between the b_i and the χ_{2n} can be easily obtained by a straightforward calculation. We report them here for completeness:

$$\chi_4 = -3! \frac{b_3}{b_1^4} \tag{3.8}$$

$$\chi_6 = -5! \left(\frac{b_5}{b_1^6} - 3 \frac{b_3^2}{b_1^7} \right) \tag{3.9}$$

$$\chi_8 = -7! \left(\frac{b_7}{b_1^8} + 12 \frac{b_3^3}{b_1^{10}} - 8 \frac{b_3 b_5}{b_1^9} \right) \tag{3.10}$$

$$\chi_{10} = -9! \left(\frac{b_9}{b_1^{10}} - 55 \frac{b_3^4}{b_1^{13}} - 10 \frac{b_3 b_7}{b_1^{11}} + 55 \frac{b_3^2 b_5}{b_1^{12}} - 5 \frac{b_5^2}{b_1^{11}} \right) \tag{3.11}$$

$$\chi_{12} = -11! \left(\frac{b_{11}}{b_1^{12}} + \frac{273 \, b_3^5}{b_1^{16}} - \frac{364 \, b_3^3 b_5}{b_1^{15}} + \frac{78 \, b_3 b_5^2}{b_1^{14}} + \frac{78 \, b_3^2 b_7}{b_1^{14}} - \frac{12 \, b_5 b_7}{b_1^{13}} - \frac{12 \, b_3 b_9}{b_1^{13}} \right). \tag{3.12}$$

This last step is the only difference in the present calculation with respect to the analogous one reported in [5], where we directly studied the thermodynamic limit of the b_i coefficients. The reason is that as n increases the χ_{2n} show a much better and smoother FSS behaviour than the b_i.

3.1.2. The thermodynamic limit. In the second step of the analysis we extrapolate our results to the thermodynamic limit. We know on general grounds that, since $L_1 \gg \xi$, the convergence towards the thermodynamic limit of the χ_{2n} functions is exponentially fast, the decay rates being related to the spectrum of the theory. This result holds for any lattice model. However, in the case of the two-dimensional Ising model, thanks to the fact that the model can be solved exactly also on finite lattices [12] (or, in the language of S-matrix theory, thanks to the fact that the S-matrix of the model is very simple), much more information can be obtained on the FSS properties of the free energy and of its derivatives. In particular, one can explicitly compute the functional form of the FSS behaviour of χ_{2n} (see appendix B). It turns out to be

$$\chi_{2n}(L_1) - \chi_{2n}(\infty) = L_1^{2n-3/2}(g_{2n,1}(1/L_1)e^{-mL_1} \\ + g_{2n,2}(1/L_1)e^{-2mL_1} + g_{2n,3}(1/L_1)e^{-3mL_1} + \cdots) \tag{3.13}$$

where $g_{2n,i}(1/L_1)$ are functions of $1/L_1$, which can be expanded in (positive) powers of $1/L_1$.

By fitting our values of $\chi_{2n}(L_1)$ with this law (by expanding the functions $g_{2n,i}(L_1)$ and using as free parameters the coefficients of the first few terms of the expansion) it is possible to obtain very precise estimates for the thermodynamic limit $\chi_{2n}(\infty)$. However, this is not the most efficient strategy, since much of the information contained in the $\chi_{2n}(L_1)$ is lost in the determination of the coefficients of the Taylor expansion of the $g_{2n,i}(L_1)$ functions. Thus, we have elaborated an alternative iterative procedure which is much simpler to perform than the above multiple fits and allows us to reach higher precision (up to one additional accurate digit). The idea behind this iterative algorithm (IA, in the following) is first to absorb the

Table 1. Thermodynamic-limit results for $\chi_n(t)\, t^{15n/8-2}$, for $n = 4, 6$. The quoted error bars are estimates of the systematic error of the extrapolation.

β	4	6
0.20	$-3.211\,114\,9829(1)$	$61.945\,049\,68(2)$
0.25	$-3.472\,139\,4468(1)$	$74.543\,748\,8(1)$
0.28	$-3.622\,540\,033(1)$	$82.255\,117\,(3)$
0.30	$-3.720\,514\,859(2)$	$87.459\,547\,(5)$
0.31	$-3.768\,883\,189(5)$	$90.081\,633\,(5)$
0.32	$-3.816\,873\,86(2)$	$92.717\,55(1)$
0.33	$-3.864\,515\,67(5)$	$95.367\,88(5)$
0.335	$-3.888\,215\,30(5)$	$96.698\,6(1)$
0.34	$-3.911\,839\,45(10)$	$98.033\,2(1)$
0.345	$-3.935\,392\,2(1)$	$99.371\,8(1)$
0.35	$-3.958\,878\,0(5)$	$100.714\,2(2)$
0.355	$-3.982\,300\,5(10)$	$102.061\,0(2)$
0.36	$-4.005\,665\,3(10)$	$103.411\,9(2)$
0.365	$-4.028\,975\,(1)$	$104.767\,3(2)$
0.37	$-4.052\,238\,(1)$	$106.127(1)$

pre-exponential factors of equation (3.13) in the masses by allowing them to depend on L_1. Then, the resulting exponential corrections are eliminated by iteratively solving the system of equations

$$\chi_{2n}^{(i)}(L_1 - 2) = c\,\exp(-x(L_1 - 2)) + \chi_{2n}^{(i+1)}(L_1)$$
$$\chi_{2n}^{(i)}(L_1 - 1) = c\,\exp(-x(L_1 - 1)) + \chi_{2n}^{(i+1)}(L_1) \qquad (3.14)$$
$$\chi_{2n}^{(i)}(L_1) \quad\;\; = c\,\exp(-xL_1) \qquad\quad + \chi_{2n}^{(i+1)}(L_1)$$

with respect to $\chi_{2n}^{(i+1)}(L_1)$, c and x.

The index i in $\chi_{2n}^{(i)}$ denotes the order of the iteration, in particular the $\chi_{2n}^{(0)}$ are the input data of the algorithm obtained as discussed in the previous section. As i increases the L_1 dependence of the $\chi_{2n}^{(i)}$ becomes smaller. After a certain number of steps the extrapolation becomes unstable since rounding errors accumulate. Typically, the final estimate was obtained by iterating 1, 2, 3 or 4 times, depending on β and on n. The residual dependence on L_1 is used to estimate the error in the best estimate of $\chi_{2n}(\infty)$. The results obtained in this way are in perfect agreement with those obtained by directly fitting equation (3.13) but, as mentioned above, turn out to be more precise. By using the IA we were able to obtain reliable estimates of χ_n for $n \leqslant 12$ up to $\beta = 0.37$. Our results are summarized in tables 1 and 2.

In order to verify whether our method of estimating the errors with the IA method (quoted in tables 1 and 2) is reliable we made the following test. We studied with the IA a sample of data at a very low value of β where rather large values of L_1/ξ could be reached, so that the results for the $\chi_{2n}(L_1)$ on our largest lattices were already good estimates of $\chi_{2n}(\infty)$ with no need of further manipulations. Then, we performed our analysis using only the data at small L_1. In this way we could explicitly check that our extrapolation scheme gives accurate results for the thermodynamic limit starting from data such that $L_1/\xi \approx 7$, which is what we reached at our largest value of β, $\beta = 0.37$. Moreover in appendix (B.3) we report a further test of IA. We applied it to a test function of the type (3.13), finding again support for the reliability of the method.

In order to give a feeling of the performances of the IA for $\beta = 0.37$—this is the value of β that is nearer to the critical point among those we studied and hence the worst one

Table 2. Thermodynamic-limit results for $\chi_n(t) \, t^{15n/8-2}$ for $n = 8, 10, 12$. The quoted error bars are estimates of the systematic error of the extrapolation.

β	8	10		12	
0.20	$-2985.5224(2)$	267 883.	(1.)	$-23\,211\,300.$	(1000.)
0.25	$-4002.0604(2)$	400 060.	(1.)	$-43\,255\,500.$	(1000.)
0.28	$-4672.309(2)$	494 203.	(5.)	$-59\,550\,000.$	(10 000.)
0.30	$-5144.20(2)$	563 455.	(10.)	$-72\,400\,000.$	(10 000.)
0.31	$-5387.715(10)$	600 090.	(20.)	$-794\,45\,000.$	(30 000.)
0.32	$-5636.31(2)$	638 110.	(50.)	$-86\,910\,000.$	(100 000.)
0.33	$-5890.10(5)$	677 600.	(100.)	$-94\,830\,000.$	(100 000.)
0.335	$-6018.9(1)$	697 800.	(100.)	$-98\,950\,000.$	(200 000.)
0.34	$-6149.0(1)$	718 300.	(100.)	$-103\,190\,000.$	(200 000.)
0.345	$-6280.5(2)$	739 000.	(500.)	$-107\,560\,000.$	(200 000.)
0.35	$-6413.1(3)$	760 600.	(200.)	$-112\,000\,000.$	(500 000.)
0.355	$-6547.1(3)$	782 400.	(200.)	$-116\,600\,000.$	(500 000.)
0.36	$-6682.3(5)$	804 400.	(300.)	$-121\,300\,000.$	(1000 000.)
0.365	$-6819.3(3)$	827 100.	(500.)	$-126\,100\,000.$	(1000 000.)
0.37	$-6958.1(5)$	850 000.	(500.)	$-131\,000\,000.$	(1000 000.)

Table 3. Results of the IA analysis for $F_4 \equiv \chi_4 t^{-11/2}$ at $\beta = 0.37$.

L_1	$F_4^{(0)}$	$F_4^{(1)}$	$F_4^{(2)}$	$F_4^{(3)}$	$F_4^{(4)}$
15	$-3.726\,094\,189\,895\,19$	$-4.074\,556\,55$			
16	$-3.803\,801\,470\,541\,99$	$-4.065\,094\,98$			
17	$-3.863\,412\,992\,924\,73$	$-4.059\,786\,84$	$-4.053\,002\,96$		
18	$-3.908\,976\,248\,238\,99$	$-4.056\,753\,21$	$-4.052\,707\,13$		
19	$-3.943\,703\,002\,436\,99$	$-4.054\,988\,69$	$-4.052\,535\,35$	$-4.052\,297\,50$	
20	$-3.970\,111\,049\,142\,28$	$-4.053\,944\,38$	$-4.052\,430\,12$	$-4.052\,263\,71$	
21	$-3.990\,156\,818\,700\,86$	$-4.053\,315\,48$	$-4.052\,363\,36$	$-4.052\,247\,54$	$-4.052\,232\,70$
22	$-4.005\,350\,649\,549\,57$	$-4.052\,930\,08$	$-4.052\,320\,14$	$-4.052\,240\,75$	$-4.052\,235\,85$
23	$-4.016\,852\,788\,478\,50$	$-4.052\,689\,80$	$-4.052\,291\,91$	$-4.052\,238\,74$	$-4.052\,237\,89$
24	$-4.025\,551\,193\,514\,43$	$-4.052\,537\,44$	$-4.052\,273\,42$	$-4.052\,238\,37$	$-4.052\,238\,29$

from the point of view of FSS—we have reported in table 3 the results of the first four iterations for χ_4 at $\beta = 0.37$. From these numbers we estimate the thermodynamic limit of χ_4 to be $\chi_4(\infty) = -4.052\,238(1)$, to be compared with the result obtained by directly fitting equation (3.13), which is $\chi_4(\infty) = -4.052\,242(10)$.

This same IA was also used in our previous paper [5], the only difference being that in that case it was applied to extract the thermodynamic limit of the b_i constants. Let us stress, as a final remark, that this algorithm is very general and can be used even when no exact information is available on the FSS behaviour of the quantity of interest except from the fact that it is dominated by an exponential decay.

3.2. Small-t expansion of $\chi_n(t)$

As well known, the TM approach gives reliable results only for rather small values of β. In order to perform the extrapolation $\beta \to \beta_c$, it is thus mandatory to have a good control of the scaling corrections of $\chi_n(t)$. In this section we address this problem in detail. Our goal will be to obtain for each χ_{2n} the exact form of the scaling function up to $O(t^4)$ and the spectrum of the possible scaling dimension (i.e. the exponents of the corresponding contributions in the

scaling function) up to $O(t^5)$.

The most important information which is needed to perform this analysis is the spectrum of the irrelevant operators of the theory. This problem was recently addressed numerically in [13–15], where it was shown that, for rotationally invariant quantities such as the free energy and its derivatives, the first correction due to the irrelevant operators in the square-lattice Ising model appears at order t^4 (hence no correction of order t^2 is present) and that the next irrelevant operator contributes only at order t^6. This result has been shown for the susceptibility in zero field, but a standard application of RG ideas implies that it should also apply to the free energy as a function[5] of t and h. This is also confirmed by the less precise numerical results of [4] for the free energy on the critical isotherm and by the analytic study of the two-point function at large distances [16].

This result allows us to determine the scaling corrections to $\chi_{2n}(t)$ up to order t^4 by using standard RG techniques (see [5, 17, 18]).

As a first step, let us write the free energy of the model in terms of nonlinear scaling fields [19]:

$$F(t, h) = F_{\mathrm{b}}(t, h) + |u_t|^{d/y_t} f_{\mathrm{sing}} \left(\frac{u_h}{|u_t|^{y_h/y_t}}, \left\{ \frac{u_j}{|u_t|^{y_j/y_t}} \right\} \right)$$

$$+ |u_t|^{d/y_t} \log |u_t| \tilde{f}_{\mathrm{sing}} \left(\frac{u_h}{|u_t|^{y_h/y_t}}, \left\{ \frac{u_j}{|u_t|^{y_j/y_t}} \right\} \right). \tag{3.15}$$

Here $F_{\mathrm{b}}(t, h)$ is a regular function of t and h^2, u_t, u_h and $\{u_j\}$ are the nonlinear scaling fields associated respectively with the temperature, the magnetic field and the irrelevant operators and y_t, y_h and $\{y_j\}$ are the corresponding dimensions. For the Ising model $y_t = 1$ and $y_h = 15/8$. Notice the presence of the logarithmic term, that is related to a 'resonance' between the thermal and the identity operator[6]. Since all numerical data indicate that for $t \to 0$ all zero-momentum correlation functions diverge as a power of t without logarithms—our results provide additional evidence for this cancellation—we shall assume in the following (as in [17]) that $\tilde{f}_{\mathrm{sing}}$ does not depend on its first argument $u_h/|u_t|^{y_h/y_t}$. There is also some evidence that the leading contribution due to the irrelevant operators is absent. Indeed, for the susceptibility one would expect a correction of order $t^4 \log |t|$, which is not found in the high-precision study of the susceptibility reported in [15]. In the following we will be conservative and we will report results with and without corrections of order $t^4 \log |t|$. Our final estimates assume however that such a term is absent.

The scaling fields are analytic functions of t and h that respect the Z_2 parity of h. Let us write their Taylor expansion, keeping only those terms that are needed for our analysis (we use the notations of [17]):

$$u_h = h \left[1 + c_h t + d_h t^2 + e_h h^2 + f_h t^3 + O(t^4, th^2) \right] \tag{3.16}$$

$$u_t = t + b_t h^2 + c_t t^2 + d_t t^3 + e_t th^2 + f_t t^4 + O(t^5, t^2 h^2, h^4). \tag{3.17}$$

All these coefficients (except e_h) have been determined exactly or numerically with very high precision (see [5, 14, 18]). We list them here for completeness:

$$c_h = \frac{\beta_{\mathrm{c}}}{\sqrt{2}} \qquad d_h = \frac{23\beta_{\mathrm{c}}^2}{16} \qquad f_h = \frac{191\beta_{\mathrm{c}}^3}{48\sqrt{2}} \tag{3.18}$$

[5] This pattern agrees with an independent analysis, in the framework of CFT, based on the spectrum of the quasi-primary operators of the critical Ising model. This result was already anticipated in [5] and will be discussed in full detail in a forthcoming publication.

[6] In principle, other logarithmic terms may arise from additional resonances due to the fact that y_j are integers or differ by integers from y_h, and indeed they have been observed in a high-precision analysis of the asymptotic behaviour of the susceptibility [15]. They will not be considered here since these contributions either are subleading with respect to those we are interested in or have a form that is already included.

$$c_t = \frac{\beta_c}{\sqrt{2}} \qquad d_t = \frac{7\beta_c^2}{6} \qquad f_t = \frac{17\beta_c^3}{6\sqrt{2}} \tag{3.19}$$

$$e_t = b_t \beta_c \sqrt{2} \qquad b_t = -\frac{E_0 \pi}{16\beta_c^2} \tag{3.20}$$

where [20]

$$E_0 = 0.040\,325\,5003\ldots \tag{3.21}$$

is the coefficient of the contribution proportional to $t \log |t|$ in the susceptibility.

The coefficient e_h is the only term that was not reported in [5]. In appendix C we show that it is possible to determine it by analysing the scaling corrections to the free energy on the critical isotherm. Using the precise data of [4], we obtain

$$e_h = -0.007\,27(15). \tag{3.22}$$

Using equation (3.15) and the expansions of the scaling fields reported above, we can compute the leading terms in the asymptotic expansion of $\chi_n(t)$ for $t \to 0$. We obtain

$$\chi_n(t) = t^{2-15n/8} a_n(t) + t^{19/4-15n/8} b_n(t) + \cdots \tag{3.23}$$

where

$$a_n(t) = C_n^+ (1 + \alpha_1 \beta_c t + \alpha_2 \beta_c^2 t^2 + \alpha_3 \beta_c^3 t^3 + O(t^4)) \tag{3.24}$$

$$b_n(t) = C_{n-2}^+ (\zeta_0 + \zeta_1 t + O(t^2)). \tag{3.25}$$

The coefficients α_i and ζ_i are given by

$$\alpha_1 = -\frac{7n - 16}{8\sqrt{2}} \tag{3.26}$$

$$\alpha_2 = \frac{147n^2 - 1080n + 2176}{768} \tag{3.27}$$

$$\alpha_3 = -\frac{343n^3 - 5208n^2 + 26\,240n - 49\,152}{6144\sqrt{2}} \tag{3.28}$$

$$\zeta_0 = \frac{E_0 n(n-1)(15n-46)\pi}{128\beta_c^2} \tag{3.29}$$

$$\zeta_1 = -\frac{E_0 n(n-1)(7n-38)(15n-46)\pi}{1024\sqrt{2}\beta_c} + n(n-2)(n-1)e_h. \tag{3.30}$$

Plugging these coefficients into equation (3.23) we obtain the exact form of the scaling function up to the contribution of the first irrelevant field, i.e. up to $O(t^4)$. The simplest way to use the exact knowledge of these terms of the scaling function to extract the amplitudes C_n^+ is to construct the quantity

$$M^{(n)}(t) \equiv \frac{\chi_n(t)\, t^{15n/8-2} - b_n(t) t^{15n/8-19/4}}{a_n(t)} \tag{3.31}$$

which has the following expansion for $t \to 0$:

$$M^{(n)}(t) = C_n^+ (1 + p_1 t^4 + \tilde{p}_1 t^4 \log t + p_2 t^{4.75} + O(t^5)). \tag{3.32}$$

The terms proportional to t^4 and to $t^{4.75}$ are due to the first unknown coefficients of $a_n(t)$ and $b_n(t)$ respectively. They also take into account the possible presence of irrelevant operators contributing to order t^4. The term proportional to $t^4 \log t$ is the only remnant of the $\tilde{f}_{\text{sing}}$ term in equation (3.15) and it is due to the irrelevant operators in $\tilde{f}_{\text{sing}}$.

The constants C_n^+ are determined in sequence. We start by using C_2^+, which is known to very high precision, $C_2^+ = 0.962\,581\,732\,308\,772\,114\,0443\ldots$ [3, 14, 15], to estimate C_4^+ with

a best-fit analysis of equation (3.32) with $n = 4$. The value of C_4^+ obtained in this way is then used as input to construct the function $b_6(t)$ which appears in equation (3.31), thus allowing us to estimate $M^{(6)}(t)$. At this point, by using again equation (3.32), we obtain the best-fit value for C_6^+ and repeat the whole procedure for the next value of n.

As a consequence, in the determination of $M^{(n)}(t)$ there are three sources of uncertainty:

(a) The uncertainty in the TM estimates of $\chi_n(t)$ which we use as input of our analysis.

(b) The uncertainty in the estimate of e_h and E_0.

(c) The uncertainty in our best estimate of C_{n-2}^+.

These uncertainties must be treated in different ways. (a) can be straightforwardly propagated to $M^{(n)}(t)$. In contrast (b) and (c) appear as uncertainties of the whole fitting function. To deal with them we followed the simplest (and most conservative) strategy. Let us study as an example the uncertainty due to the error on e_h. We constructed two sets of data, $M_+^{(n)}(t)$ and $M_-^{(n)}(t)$, obtained by using $e_h + \delta e_h$ and $e_h - \delta e_h$ respectively, and then, for each set, we determined the best-fit values of the parameters. The difference between these results is a conservative estimate of the error induced by the uncertainty on e_h. In order to give an idea of the size of these uncertainties, we have reported in table 5 those induced by e_h in the best-fit estimates of C_6^+.

It turns out that the errors induced by the uncertainties on C_{n-2}^+ and E_0 are always negligible with respect to that due to e_h and, what is more important, that this last one is in any case negligible with respect to the uncertainty in the best-fit estimate, i.e. with respect to the uncertainty due to the lack of knowledge of the higher-order correction terms. For this reason we shall neglect the errors of type (b) and (c) in our final results.

3.3. The fitting procedure

To analyse the data we followed a procedure similar to that presented in [5]. For each set of data we performed five different fits of $M^{(n)}$ using equation (3.32),

(f1) keeping C_n^+ and p_1 as free parameters, and setting $\tilde{p}_1 = p_2 = 0$;

(f2) keeping C_n^+ and $\tilde{p}_1$ as free parameters, and setting $p_1 = p_2 = 0$;

(f3) keeping C_n^+, p_1, and $\tilde{p}_1$ as free parameters, and setting $p_2 = 0$;

(f4) keeping C_n^+, p_1, and p_2 as free parameters, and setting $\tilde{p}_1 = 0$;

(f5) keeping C_n^+, p_1, $\tilde{p}_1$ and p_2 as free parameters.

In order to estimate the systematic errors involved in the determination of C_n^+, we performed for all the fitting functions several independent fits, trying first to fit all the existing data (reported in tables 1 and 2) and then eliminating the data one by one, starting from the farthest from the critical point. Among the set of estimates of the critical amplitudes we selected only those fulfilling the following requirements:

(1) The reduced χ^2 of the fit must be of order unity. In order to fix precisely a threshold, we required the fit to have a confidence level larger than 30%.

(2) For all the subleading terms included in the fitting function, the amplitude estimated from the fit must be larger than the corresponding error.

(3) If the fit contains k free parameters besides C_n^+, then at least 2^k degrees of freedom must be used in the fit. This means that in fits of type (f1) and (f2) at least four data must be used ($k = 1$: two parameters plus two degrees of freedom); in those of type (f3) and (f4) at

Table 4. Best-fit estimates of the amplitudes C_n^+.

	$\tilde{p}_1 \neq 0$	$\tilde{p}_1 = 0$
C_4^+	$-4.379\,095(8)$	$-4.379\,094(6)$
C_6^+	$125.9330(11)$	$125.9332(6)$
C_8^+	$-9066.5(9)$	$-9066.4(7)$
C_{10}^+	$1216.33(80) \times 10^3$	$1216.34(60) \times 10^3$
C_{12}^+	$-262\,60(150) \times 10^4$	$-261\,75(60) \times 10^4$

Table 5. Estimates of C_6^+. In the first column we report the best-fit results for the critical amplitude (the first error in parenthesis is that induced by the systematic errors of the input data while the second is due to the error on e_h), in the second column the number of degrees of freedom (i.e. the number of data used in the fit minus the number of free parameters) and in the last column the type of fit.

C_6^+	d.o.f.	Fit type
$125.932\,74(12)(0)$	2	(f1)
$125.932\,98(5)(0)$	9	(f1)
$125.933\,01(14)(2)$	2	(f2)
$125.933\,89(16)(7)$	7	(f2)
$125.932\,23(30)(1)$	5	(f3)
$125.933\,41(15)(2)$	8	(f3)
$125.932\,90(14)(0)$	7	(f4)
$125.933\,69(12)(1)$	9	(f4)
$125.932\,53(20)(2)$	8	(f5)
$125.932\,58(7)(2)$	10	(f5)

least seven data must be used ($k = 2$: three parameters plus four degrees of freedom). In (f5) at least 12 data must be used ($k = 3$: four parameters plus eight degrees of freedom).

This last constraint is a generalization to higher values of n of the one that we proposed in [5]. It seems to encode very well the real statistical significance of the data.

Finally, among all the estimates of the critical amplitude C_n^+ fulfilling these requirements we selected the smallest and the largest ones as lower and upper bounds. We consider the mean of these two values as our best prediction for C_n^+. The values are reported in table 4. The errors quoted in table 4 are half of the difference between the upper and lower bounds. They seem to give a reliable estimate of the uncertainty of our results.

Let us briefly comment on these results:

 (a) In table 4 we have reported in two separate columns the best-fit results with and without the log-type contribution $\tilde{p}_1 t^4 \log t$. The reason for this choice is that there are strong numerical indications [15] that $\tilde{p}_1 = 0$ in the Ising model. To keep our analysis as general as possible, we report also the result with $\tilde{p}_1 \neq 0$. It is interesting to note that the estimates of C_n^+ ($n \leqslant 10$) do not depend essentially on this choice, while the error decreases by a factor of 1.5–2 if we assume $\tilde{p}_1 = 0$. This seems to indicate that the fitting procedure is stable and reliable.
 (b) In the first line of table 4 we report our best-fit estimate of C_4^+, which we had already estimated in [5] with the same techniques as used in the present paper. The only improvement with respect to [5] is that we now also know exactly the value of e_h, which was the dominant (of order $t^{3.75}$) unknown correction in [5]. It is instructive to compare

the two estimates. In [5] we obtained, taking also into account the log-type corrections, $C_4^+ = -4.379\,101(9)$, while the present value is $C_4^+ = -4.379\,095(8)$. The two results are, as expected, fully compatible. The new one is only slightly more precise than the previous one. The fact that the enhancement in precision is so small is due to the fact that, by using the exact value of e_h, we only improve the known small-t expansion of $\chi_n(t)$ from $O(t^{3.75})$ to $O(t^4)$.

(c) It is instructive to look in more detail at the results of the fits in one particular case. Let us consider, for instance, C_6^+. The results are reported in table 5, where, for each type of fit, we quote the two fits which correspond to the highest and lowest values of C_6^+ (in quoting the uncertainties in the best-fit estimates we also report those induced by e_h, to give an idea of their size).

The most impressive feature of this set of fits is that with only the t^4 contribution besides C_6^+ (fit (f1)) one can fit all the data up to $\beta = 0.31$.

Looking at table 5, we obtain our best estimate

$$C_6^+ = 125.9330(11). \tag{3.33}$$

If we additionally assume that $\tilde{p}_1 = 0$, i.e. we take into account only the results of the fits of type (f1) and (f4), then we find

$$C_6^+ = 125.9332(6). \tag{3.34}$$

The estimates (3.33) and (3.34) are those reported in table 4.

3.4. Summary of the results

Using the estimates of C_{2n}^+ obtained in the previous section, we can estimate r_{2n} by using equations (3.2)–(3.5). We finally obtain

$$g_4 = 14.697\,323(20) \tag{3.35}$$
$$r_6 = 3.678\,66(3)(2) \tag{3.36}$$
$$r_8 = 26.041(8)(3) \tag{3.37}$$
$$r_{10} = 284.5(1.4)(1.0) \tag{3.38}$$
$$r_{12} = 4200(320)(420). \tag{3.39}$$

For each r_{2n} we report two errors: the first one is due to the error on C_{2n}^+, while the second one expresses the uncertainty due to the error on all $C_{2k}^+, k < n$. Note that the high precision reached in the estimates of C_{2n}^+ is partially lost in the estimates of r_{2n} because of the cancellations among the various terms in the sums. This effect is particularly important for r_{10} and r_{12}.

Tables 6 and 7 compare our results with the existing data from other approaches. Our estimates of r_{2n} perfectly agree with those obtained analysing HT expansions. In contrast, the Monte Carlo results for r_{2n} of [25] are systematically larger. This could be an indication that the finite-size scaling curves obtained in [25] are still affected by large scaling corrections.

4. The critical equation of state

4.1. General features

The basic result of the RG theory is that asymptotically close to the critical point the equation of state may be written in the scaling form [32]

$$h = \frac{\partial \mathcal{F}}{\partial M} \propto M|M|^{\delta-1} f(x) \qquad x \propto t|M|^{-1/\beta} \tag{4.1}$$

Table 6. Estimates of g_4. We also report the existing results from the form-factor (FF) approach, HT expansions, Monte Carlo (MC) simulations, field theory (FT) based on the ϵ expansion and the fixed-dimension $d = 2$ g expansion, and a method based on a dimensional expansion around $d = 0$ (d-expansion).

Method	Ref.	g_4
TM + CFT	(this work)	14.697 323(20)
FF	[21]	14.697 5(1)
HT	[22]	14.694 (2)
HT	[23]	14.693 (4)
HT	[24]	14.700 (17)
MC	[21]	14.69(2)
MC	[25]	14.7(2)
FT ϵ expansion	[26]	14.7(4)
FT g expansion	[27]	15.5(8)
FT g expansion	[28]	15.4(3)
d-expansion	[29]	14.88(17)

Table 7. Estimates of r_{2n}. We also report the existing results from HT expansions, Monte Carlo (MC) simulations and field theory (FT) based on the ϵ expansion and the fixed-dimension $d = 2$ g expansion.

	TM + CFT	HT [30]	HT [24]	MC [25]	FT ϵ-exp [30]	FT g-exp [31]
r_6	3.678 66(5)	3.678(2)	3.679(8)	3.93(12)	3.69(4)	3.68
r_8	26.041(11)	26.0(2)		28.0(1.6)	26.4(1.0)	
r_{10}	284.5(2.4)	275(15)				
r_{12}	4200(740)					

where $f(x)$ is a universal scaling function normalized in such a way that $f(-1) = 0$ and $f(0) = 1$. The value $x = -1$ corresponds to the coexistence curve, and $x = 0$ to the critical point $t = 0$. The function $h(M, t)$, representing the external field in the critical equation of state, satisfies Griffiths' analyticity, i.e. it is regular at $t = 0$ for $M > 0$ fixed and at $M = 0$ for $t > 0$ fixed. This implies that $f(x)$ is analytic at $x = 0$, and it has a regular expansion for large x of the form

$$f(x) = x^\gamma \sum_{n=0}^{\infty} f_n^\infty x^{-2n\beta}. \tag{4.2}$$

As already mentioned in the introduction, many things are exactly known for the two-dimensional Ising model. However, there is no exact expression for the free energy and for the critical equation of state in the whole (t, h) plane. In table 8 we report a summary of the known results for the two-dimensional Ising model (there we consider only infinite-volume quantities). Many of them are known exactly; for the others we report the best estimate. The results that have not been derived in this paper have been taken from [5, 33–36].

In the following we shall determine the equation of state, starting from its expansion for small magnetization in the HT phase. It is therefore useful to introduce a different representation that is analytic for $M \to 0$. Using the results of sections 2 and 3.2, in particular equation (3.15) and the discussion following it, one may write the Helmholtz free energy as

$$\mathcal{F}_{\text{sing}}(t, M) = at^2 V(z) + \frac{A}{2}t^2 \log |t| \tag{4.3}$$

Table 8. Critical exponents and universal amplitude ratios for the two-dimensional Ising universality class. See appendix A for the definitions.

Critical exponents and amplitude ratios	
γ	7/4
ν	1
η	1/4
β	1/8
δ	15
ω	2
$U_0 \equiv A^+/A^-$	1
$U_2 \equiv C_2^+/C_2^-$	37.693 652 01
$R_c^+ \equiv A^+ C_2^+/B^2$	0.318 569 39
$R_c^- \equiv A^- C_2^-/B^2$	0.008 451 54
$R_\chi \equiv Q_1^{-\delta} \equiv C_2^+ B^{\delta-1}/(\delta C^c)^\delta$	6.778 285 02
$w^2 \equiv C_2^-/[B^2(f^-)^2]$	0.531 526 07
$U_\xi \equiv f^+/f^-$	3.162 495 04
$U_{\xi_{\text{gap}}} \equiv f_{\text{gap}}^+/f_{\text{gap}}^-$	2
$Q^+ \equiv A^+(f^+)^2$	0.159 027 04
$Q^- \equiv A^-(f^-)^2$	0.015 900 517
$Q_\xi^+ \equiv f_{\text{gap}}^+/f^+$	1.000 402 074
$Q_\xi^c \equiv f_{\text{gap}}^c/f^c$	1.078 6828
$Q_\xi^- \equiv f_{\text{gap}}^-/f^-$	1.581 883 299
$Q_2 \equiv (f^c/f^+)^{2-\eta} C_2^+/C^c$	2.835 5305
$g_4 \equiv -C_4^+/[(C_2^+)^2(f^+)^2]$	14.697 323(20)
$R_4^+ \equiv -C_4^+ B^2/(C_2^+)^3$	7.336 774(10)
$r_6 \equiv g_6/g_4^2$	3.678 66(5)
$r_8 \equiv g_8/g_4^3$	26.041(11)
$r_{10} \equiv g_{10}/g_4^4$	284.5(2.4)
$r_{12} \equiv g_{12}/g_4^5$	$4.44(6) \times 10^3$
$r_{14} \equiv g_{14}/g_4^6$	$8.43(3) \times 10^5$
$v_3 \equiv -C_3^- B/(C_2^-)^2$	33.011(6)
$v_4 \equiv -C_4^- B^2/(C_2^-)^3 + 3v_3^2$	48.6(1.2)

where

$$z = b|M|t^{-\beta} \qquad V(z) = \frac{z^2}{2} + \frac{z^4}{4!} + O(z^6)$$

$$a = -(C^+)^2/C_4^+ \qquad b = \left[-\frac{C_4^+}{(C^+)^3}\right]^{1/2}.$$

(4.4)

The constant A is related to the amplitudes of the specific heat for $h \to 0$ defined in equation (A.1). Indeed the analyticity of the free energy for $t = 0$, $h \neq 0$, implies $A^+ = A^- \equiv A$, and thus $U_0 \equiv A^+/A^- = 1$. The presence of the logarithmic term gives rise to logarithms in the expansions of $V(z)$ for $z \to \infty$. Indeed, the analyticity of $\mathcal{F}_{\text{sing}}(t, M)$ for $t = 0$, $|M| \neq 0$ implies, for large z,

$$V(z) = z^{16} \sum_{k=0} c_k z^{-8k} + c_{\log} \log z.$$

(4.5)

The constant $c_{\log}$ is easily expressed in terms of invariant amplitude ratios:

$$c_{\log} = \frac{4A}{a} = 4Q^+ g_4^+.$$

(4.6)

For the equation of state we have

$$h = \frac{\partial \mathcal{F}}{\partial M} = a\,b\,t^{15/8}\frac{\partial V(z)}{\partial z} \equiv a\,b\,t^{15/8}B(z) \tag{4.7}$$

where, using equation (2.8),

$$B(z) = z + \frac{1}{6}z^3 + \sum_{j=3}\frac{r_{2j}}{(2j-1)!}z^{2j-1}. \tag{4.8}$$

For large z, using (4.5) we obtain the expansion

$$B(z) = z^{15}\sum_{k=0}B_k^\infty z^{-8k}. \tag{4.9}$$

The constant B_0^∞ can be expressed in terms of invariant amplitude ratios:

$$B_0^\infty = R_\chi(R_4^+)^{(1-\delta)/2} = 0.592\,357(6) \times 10^{-5} \tag{4.10}$$

where we have used the numerical results of table 8. Moreover, by using equation (4.5), we obtain

$$B_2^\infty = c_{\log} = 4Q^+g_4 = 9.349\,087(13). \tag{4.11}$$

To reach the coexistence curve, corresponding to $t < 0$ and $h = 0$, one should perform an analytic continuation in the complex t-plane [6, 7]. The spontaneous magnetization is related to the complex zero $z_0 = |z_0|\mathrm{e}^{-\mathrm{i}\pi\beta}$ of $B(z)$ [7], where

$$|z_0|^2 = R_4^+ \equiv -C_4^+B^2/(C_2^+)^3 = 7.336\,774(10). \tag{4.12}$$

Therefore, the description of the coexistence curve is related to the behaviour of $B(z)$ in the neighbourhood of z_0.

The functions $B(z)$ and $f(x)$ give equivalent representations of the equation of state. Indeed, they are simply related by

$$z^{-\delta}B(z) = B_0^\infty f(x) \qquad z = |z_0|x^{-\beta}. \tag{4.13}$$

4.2. Parametric representations

In order to obtain a representation of the critical equation of state that is valid in the whole critical region, we need to extend analytically the expansion (4.8) to the low-temperature region $t < 0$. For this purpose, one may use parametric representations, which implement in a simple way all scaling and analytic properties [8–10]. One may parametrize M and t in terms of R and θ according to

$$\begin{aligned} M &= m_0R^\beta\theta \\ t &= R(1-\theta^2) \\ h &= h_0R^{\beta\delta}h(\theta) \end{aligned} \tag{4.14}$$

where h_0 and m_0 are normalization constants. The variable R is non-negative and measures the distance from the critical point in the (t, h) plane; the critical behaviour is obtained for $R \to 0$. The variable θ parametrizes the displacements along the lines of constant R. The line $\theta = 0$ corresponds to the HT phase $t > 0$ and $h = 0$, the line $\theta = 1$ to the critical isotherm $t = 0$ and $\theta = \theta_0$, where θ_0 is the smallest positive zero of $h(\theta)$, to the coexistence curve $T < T_c$ and $h \to 0$. Of course, one should have $\theta_0 > 1$. The regularity properties of the critical equation of state require $h(\theta)$ to be analytic in the physical domain $0 \leqslant \theta < \theta_0$. This is at variance with what happens for the scaling functions $f(x)$ and $B(z)$, that are nonanalytic for $x \to \infty$ and $z \to \infty$ respectively. This fact is important from a practical point of view. Indeed, in

order to obtain approximate expressions of the equation of state, one can approximate $h(\theta)$ with analytic functions. The structure of the parametric representation automatically ensures the correct analytic properties of the equation of state.

Note that the mapping (4.14) is invertible only in the region $\theta < \theta_l$, where

$$\theta_l^2 = \frac{1}{1 - 2\beta} = \frac{4}{3}. \tag{4.15}$$

Thus, the physically relevant interval $0 \leqslant \theta \leqslant \theta_0$ must be contained in the region $\theta < \theta_l$, and thus we should have $\theta_0 < \theta_l$. In practice, since θ_l is a singular point of the mapping, it is important that $\theta_l - \theta_0$ is not too small. As we shall see, all our approximations satisfy this condition.

The function $h(\theta)$ is odd in θ, and is normalized so that $h(\theta) = \theta + O(\theta^3)$. Since $M = C_2^+ht^{-\gamma}$ for $M \to 0, t > 0$, this condition implies $m_0 = C^+h_0$. Following [7], we then replace h_0 by a single normalization constant ρ in such a way that we can write

$$z = \rho \, \theta \, \left(1 - \theta^2\right)^{-\beta} \tag{4.16}$$

$$B(z(\theta)) = \rho \left(1 - \theta^2\right)^{-\beta\delta} h(\theta). \tag{4.17}$$

In the exact parametric equation the value of ρ may be chosen arbitrarily: clearly the physical function $B(z)$ does not depend on it. However, if we adopt an approximation for $h(\theta)$, as we will do, the dependence of $B(z)$ on ρ is not eliminated. One may then choose ρ to obtain an optimal approximation.

From the function $h(\theta)$ one may calculate the scaling functions $f(x)$, using the relations

$$x = \frac{1 - \theta^2}{\theta_0^2 - 1} \left(\frac{\theta}{\theta_0}\right)^{-1/\beta}$$

$$f(x) = \theta^{-\delta} \frac{h(\theta)}{h(1)} \tag{4.18}$$

and all universal amplitude ratios involving zero-momentum quantities, such as the n-point susceptibilities (see, e.g., [11] for a list of formulae).

4.3. Polynomial approximations for $h(\theta)$

In order to construct approximate parametric representations, we consider a systematic approximation scheme based on polynomial approximations of $h(\theta)$ [7], i.e.

$$h^{(k)}(\rho, \theta) = \theta + \sum_{i=1}^{k-1} h_{2i+1}(\rho)\theta^{2i+1}. \tag{4.19}$$

The coefficients $h_{2i+1}(\rho)$ are functions of ρ, γ and β, and are obtained by matching the small-z expansion of $B(z)$ to $O(z^{2k-1})$, cf equation (4.17). This kind of approximation turned out to be effective for the determination of the critical equation of state of three-dimensional Ising-like systems [7, 11], and was generalized to models with Goldstone singularities [37]. In order to optimize ρ for a given truncation $h^{(k)}(\rho, \theta)$, we use a procedure based on the physical requirement of minimal dependence on ρ of the resulting universal function

$$B^{(k)}(\rho, z) \equiv \rho \left(1 - \theta^2\right)^{-\beta\delta} h^{(k)}(\rho, \theta). \tag{4.20}$$

One may indeed prove [11] that for any truncation k there exists a solution ρ_k independent of z that satisfies a global stationarity condition, i.e.

$$\left. \frac{\partial B^{(k)}(\rho, z)}{\partial \rho} \right|_{\rho = \rho_k} = 0 \tag{4.21}$$

Table 9. Universal amplitude ratios obtained by taking different approximations of the parametric function $h(\theta)$. The reported 'errors' are only related to the uncertainty of the corresponding input parameters. Numbers marked with an asterisk are inputs, not predictions.

	$h^{(3)}$	$h^{(4)}$	$h^{(5)}$
ρ	2.065	2.027	2.018(3)
$\theta_l^2 - \theta_0^2$	0.183	0.177	0.173
r_6	*3.678 66(5)	*3.678 66(5)	*3.678 66(5)
r_8	24.413	*26.041(11)	*26.041(11)
r_{10}	249.11	277.1(2)	*284.5(2.4)
r_{12}	3513.7(2)	4066(4)	4215(49)
$B(|z_0|/2)$	1.9621	1.9666	1.9670
$B(|z_0|)$	37.160	41.655	42.808
$B(2|z_0|)$	431 786	538 946	569 182
B_0^∞	0.4225×10^{-5}	$0.5279(8) \times 10^{-5}$	$0.557(10) \times 10^{-5}$
R_4^+	7.879	7.558(2)	7.47(3)
R_χ	7.967	7.434(4)	7.23(7)
U_2	48.565	44.41(3)	42.7(6)
v_3	28.009	28.756(6)	29.2(2)

for all values of z.

As input parameters we use the coefficients of the small-z expansion of $B(z)$, i.e. the estimates of coefficients r_{2j} obtained by TM + CFT and reported in table 7.

In table 9 we report the universal amplitude ratios derived from truncations corresponding to $k = 3, 4, 5$. We will not report the results for $k = 6$ obtained using the available estimate of r_{12}, because the relatively large uncertainty on r_{12} induces a very large error in the results of the $k = 6$ truncation. We only mention that results with $k = 6$ are perfectly consistent with those obtained from the $k = 5$ truncation. This can be inferred from the fact that the estimate of r_{12} obtained using $h^{(5)}(\theta)$, $r_{12} \simeq 4215$, is very close to the central value of the TM + CFT estimate, i.e. $r_{12} = 4200(740)$. As we shall see, we will obtain a much better estimate of r_{12} in section 4.4.

The results of table 9 are not stable as k increases, showing a systematic drift up to $k = 6$, where the large uncertainty does not allow a meaningful comparison. We observe that the results for the universal amplitude ratios B_0^∞, R_4^+, R_χ, U_2 and v_3 effectively converge towards their precise estimates reported in table 8. It is also reassuring that the difference between the exact value and the estimate obtained using $h^{(5)}$ is of the order of the variation of the estimates with changing k.

4.4. Improved approximations from constrained polynomials

Although the polynomial approximations we presented in the previous section are substantially consistent with the known results for the amplitude ratios, they do not provide an accurate approximation of the equation of state. The convergence appears rather slow, probably requiring the knowledge of a considerably larger number of coefficients r_{2j} to substantially improve the results. Here, we will present an improved approximation scheme that significantly increases the precision of the results.

The approximation scheme can be improved by constructing constrained polynomial approximations of $h(\theta)$ that take into account the large-z asymptotic behaviour of $B(z)$:

$$B(z) = B_0^\infty z^\delta \left[1 + \mathrm{O}(z^{-1/\beta}) \right] \tag{4.22}$$

where the value of B_0^∞ is reported in equation (4.10). We consider constrained polynomial

Table 10. Polynomial approximations of $h(\theta)$ using the global stationarity condition for various values of the truncation parameter k, cf equation (4.23). The reported expressions correspond to the central values of the input parameters.

k	ρ_k	θ_0^2	$\bar{h}^{(k)}(\theta)/[\theta(1-\theta^2/\theta_0^2)]$
2	2.011 16	1.152 78	$1-0.208\,408\theta^2$
3	2.007 70	1.159 40	$1-0.215\,675\theta^2-0.039\,403\theta^4$
4	2.007 70	1.164 41	$1-0.219\,388\theta^2-0.041\,791\theta^4-0.013\,488\theta^6$
5	2.008 81	1.169 51	$1-0.222\,389\theta^2-0.043\,547\theta^4-0.014\,809\theta^6-0.007\,168\theta^8$

approximations of the form

$$\bar{h}^{(k)}(\rho,\theta) = \theta + \sum_{i=1}^{k-1}\bar{h}_{2i+1}(\rho)\theta^{2i+1} + \bar{h}_{2k+1}(\rho)\theta^{2k+1} \tag{4.23}$$

where the coefficients $\bar{h}_{2i+1}(\rho)$ with $i<k$ are determined as before, by matching the small-z expansion of $B(z)$ to $O(z^{2k-1})$, while $\bar{h}_{2k+1}(\rho)$ is fixed by requiring that

$$B_0^\infty = \rho^{1-\delta}\bar{h}^{(k)}(\rho,1) = 0.592\,357(6)\times 10^{-5}. \tag{4.24}$$

It follows that

$$\bar{h}_{2k+1}(\rho) = \rho^{\delta-1}B_0^\infty - 1 - \sum_{i=1}^{k-1}\bar{h}_{2i+1}(\rho). \tag{4.25}$$

In this approximation scheme the free parameter ρ can be still determined by requiring the global stationarity condition (4.21). This nontrivial property is essentially due to the fact that the constraint (4.24) is linear in the coefficients $\bar{h}_{2i+1}$. It can be proved by using arguments similar to those employed in appendix C of [11] to show the global stationarity condition (4.21) for the approximation scheme (4.19). In table 10, for $k=2,3,4,5$, we report the polynomials $\bar{h}^{(k)}(\theta)$ obtained by using the global stationarity condition to fix ρ, and the central values of the input parameters F_0^∞, r_6, r_8 and r_{10}. Note the stability of the coefficients of the polynomials with k and that the size of the coefficients decreases with the order of the polynomial. The results for some universal quantities are reported in table 11. They are in much better agreement with the exact results than those obtained without constraint.

In figure 1 we show the scaling function $B(z)$ obtained from $\bar{h}^{(k)}(\rho,\theta)$ for $k=2,3,4,5$. The convergence is good; indeed, their differences are not visible in figure 1. This allows us to determine $B(z)$ for all real $z>0$ with a relative precision of at least a few per thousand (the least precision is found around $z\simeq|z_0|\simeq2.71$). This fact is not trivial since the small-z expansion has a finite convergence radius, which is expected to be $|z_0|=(R_4^+)^{1/2}\simeq2.71$. Therefore, the determination of $B(z)$ on the whole positive real axis from its small-z expansion requires an analytic continuation, which is effectively performed by the approximate parametric representations we have considered. We recall that the large-z limit corresponds to the critical isotherm $t=0$, so positive real values of z describe the HT phase up to $t=0$. Note also the good agreement of the results for B_2^∞ (see table 11), i.e. the next-next-to-leading coefficient of the large-z expansion of $B(z)$, with the precise estimate given in equation (4.11).

The convergence of the polynomial representations at the coexistence curve is slower. This can be seen by looking at the estimates reported in table 11 for universal amplitude ratios involving quantities related to the coexistence curve, such as R_4^+, R_χ, and U_2, and comparing them with the corresponding known results reported in tables 8. They appear to (monotonically) converge toward the correct results. The rate of convergence worsens when quantities with more and more derivatives with respect to h are involved in the amplitude ratio, as can already be seen by comparing the results for R_4^+ and U_2.

Table 11. Universal amplitude ratios obtained from the constrained polynomial approximations (4.23) of the parametric function $h(\theta)$. The reported 'errors' are only related to the uncertainty of the corresponding input parameters (they are reported only if they are larger than the last figure shown). Numbers marked with an asterisk are inputs, not predictions.

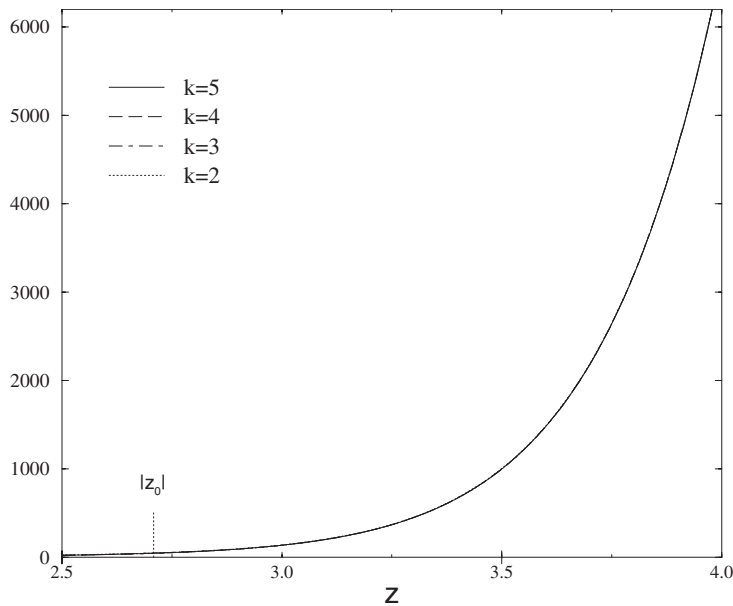|	$\bar{h}^{(2)}$	$\bar{h}^{(3)}$	$\bar{h}^{(4)}$	$\bar{h}^{(5)}$
ρ	2.011	2.008	2.008	2.009(1)
$\theta_l^2 - \theta_0^2$	0.181	0.174	0.169	0.164(2)
r_6	3.929	*3.678 66(5)	*3.678 66(5)	*3.678 66(5)
r_8	27.585	26.932	*26.041(11)	*26.041(11)
r_{10}	297.25	292.89	292.89	*284.5(2.4)
r_{12}	4425.2	4385.6	4385.6	4443(16)
r_{14}	84 387	84 029	84 029	84 305(79)
$B(|z_0|/2)$	1.9798	1.9690	1.9675	1.9672
$B(|z_0|)$	44.930	44.335	44.146(2)	44.05(3)
$B(3|z_0|/2)$	8442.2	8432.7	8429.4	8427.7(5)
$B(2|z_0|)$	604 619(6)	604 548(6)	604 524(6)	604 511(7)
$B(3|z_0|)$	$2.634\,97(3) \times 10^8$	$2.634\,96(3) \times 10^8$	$2.634\,95(3) \times 10^8$	$2.634\,95(3) \times 10^8$
B_0^∞	$*0.592\,357(6) \times 10^{-5}$	$*0.592\,357(6) \times 10^{-5}$	$*0.592\,357(6) \times 10^{-5}$	$*0.592\,357(6) \times 10^{-5}$
B_1^∞	0.021 893	0.021 375	0.021 198(3)	0.021 10(3)
B_2^∞	9.3987	9.2611	9.2611	9.286(7)
R_4^+	7.458	7.396	7.371	7.355(5)
R_χ	7.602	7.172	7.002(2)	6.90(3)
U_2	45.918	42.358	40.76(2)	39.6(3)
v_3	28.328	29.201	29.837(9)	30.5(2)

Figure 1. The scaling function $B(z)$ versus z. We report the curves obtained from the constrained polynomial approximations (4.23) with $k = 2, 3, 4, 5$. Their differences are not visible.

Figure 2 shows the scaling function $f(x)$ as obtained by the truncations $k = 2, 3, 4, 5$. The accuracy of the determination of $f(x)$ can be inferred from the convergence of the curves
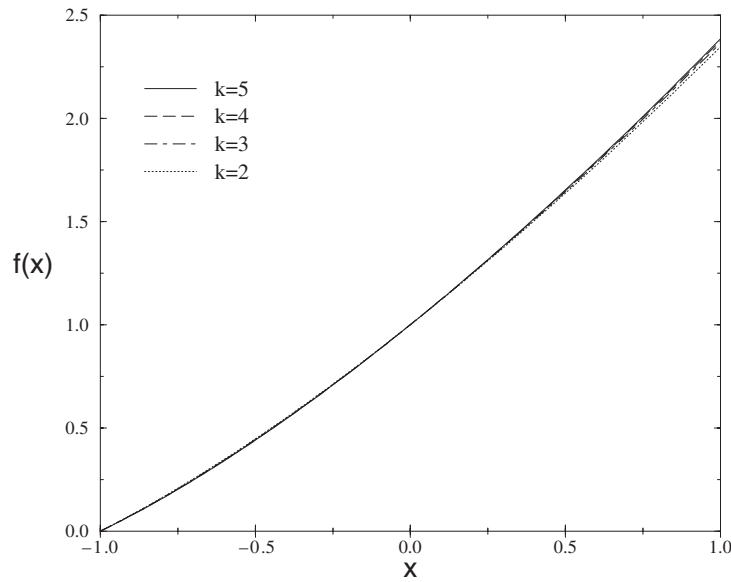
Figure 2. The scaling function $f(x)$ versus x. We report the curves obtained from the constrained polynomial approximations (4.23) with $k = 2, 3, 4, 5$.

with increasing k and the comparison with the known behaviour for $x \to -1$ and $x \to +\infty$. Indeed, for $x \to -1$ we have

$$f(x) = b_f(1 + x) + O\big[(1 + x)^2\big]$$
$$b_f = \frac{\beta U_2}{R_\chi} = 0.695\,117\,78\ldots \tag{4.26}$$

and for $x \to +\infty$

$$f(x) = f_0^\infty x^\gamma + O\big(x^{\gamma - 2\beta}\big)$$
$$f_0^\infty = R_\chi^{-1} = 0.147\,529\,94\ldots. \tag{4.27}$$

This shows that $f(x)$ is determined with a precision of a few per cent in the whole region. This happens also in the large-x region, which corresponds to the HT phase, and therefore to $z \ll 1$ in $F(z)$, essentially because $f(x)$ is normalized at the coexistence curve, i.e. $x = -1$, where our approximation is worse. For $x > 0$, the error on $f(x)$ increases from 0 to 2%, the relative error on R_χ.

Using the results of table 11 we also obtain

$$r_{12} = 4.44(6) \times 10^3 \tag{4.28}$$
$$r_{14} = 8.43(3) \times 10^5 \tag{4.29}$$
$$B_1^\infty = 0.0211(2). \tag{4.30}$$

Note that the above-reported estimate of r_{12} is perfectly consistent with the result obtained by TM + CFT, i.e. $r_{12} = 4.20(74) \times 10^3$, but much more precise.

5. Conclusions

Let us briefly summarize the main results of this paper, which presents a determination of the equation of state of the two-dimensional Ising model in the whole (t, h) plane. The starting

point of the analysis was the determination of the $2n$-point couplings constants in the HT phase, which are related to the expansion of the free energy in powers of the magnetization. These quantities were then used as input parameters for a systematic approximation scheme, which allowed us to obtain an accurate determination of the equation of state in the critical regime in the whole (t, h) plane.

The determination of the coefficients r_{2j} of the small-magnetization expansion of the Helmholtz free energy is in itself an interesting and nontrivial problem. We addressed this problem by using TM techniques. In order to overcome the typical problem of all TM calculations, i.e. the fact that they are constrained to rather small values of the lattice size in the transverse direction (in our case we could reach $L_1 = 24$ as a maximum value) we used a two-step strategy:

(1) In the first step we studied the model at very small values of β so as to have small values for ξ (whose maximum value was $\xi = 3.350\,288\ldots$, which was reached for the largest value of β that we studied: $\beta = 0.37$) and thus large values of the ratio L_1/ξ. We then extrapolated to the thermodynamic limit the TM results by using a new effective iterative algorithm. This algorithm is rather interesting in itself and could be of general utility for those working with TM methods.
(2) Second, we used a combination of standard RG results and a set of high-precision numerical information on the zero-field magnetic susceptibility to construct the scaling functions for the derivatives of the free energy involved in the construction of the r_{2j} coefficients. With these scaling functions we could obtain the critical-limit values of the r_{2j} coefficients with rather small errors even if the input data for this continuum-limit extrapolation correspond to rather small values of β.

The expansion of the free energy in powers of the magnetization in the HT phase was then used as a starting point to construct approximate representations of the equation of state that are valid in the critical regime of the whole (t, h) plane. We considered a systematic approximation scheme based on polynomial parametric representations, devised to match the known terms of the small-magnetization expansion and the large-magnetization behaviour of the Helmholtz free energy. A global stationarity condition was used in order to optimize the polynomial approximation. This approximation scheme can be improved systematically by considering higher- and higher-order polynomials. It is only limited by the number of known terms in the small-magnetization expansion of the free energy. The knowledge of this expansion up to tenth order allowed us to obtain an accurate determination of the critical equation of state. We indeed obtained the scaling function $B(z)$, cf equation (4.7), for all real $z > 0$ with a relative precision of at least a few per thousand, and the scaling function $f(x)$, cf equation (4.1), with a precision of a few per cent in the whole physical region $x \geqslant -1$. The approximation scheme is systematic, thus this precision can be improved by a more accurate knowledge of the small-magnetization expansion of the free energy in the HT phase.

The method that we used to reconstruct the equation of state from the small-magnetization expansion of the Helmholtz free energy is general and can be applied to other statistical models. We mention that similar methods have been successfully applied to the three-dimensional Ising [7, 11] and XY universality [37, 38] classes, leading to accurate determinations of the critical equation of state and of the universal ratios of amplitudes that can be extracted from it.

Acknowledgment

Appendix A. Notations for the amplitudes

Universal amplitude ratios characterize the critical behaviour of thermodynamic quantities that do not depend on the normalizations of the external (e.g. magnetic) field, order parameter (e.g. magnetization) and temperature. Amplitude ratios of zero-momentum quantities can be derived from the critical equation of state. Beside the amplitudes C_n^+ of the n-point functions $\chi_n(t)$ defined in equation (3.1) and the corresponding ones C_n^- in the low-temperature phase, we consider amplitudes derived from the singular behaviour of the specific heat

$$C_H = A^{\pm} \ln(1/|t|) \tag{A.1}$$

and the spontaneous magnetization on the coexistence curve

$$M = B(-t)^{\beta}. \tag{A.2}$$

We complete our list of amplitudes by considering the second-moment correlation length

$$\xi_{2nd} = f^{\pm}|t|^{-\nu} \tag{A.3}$$

and the true (on-shell) correlation length, describing the large-distance behaviour of the two-point function,

$$\xi_{gap} = f_{gap}^{\pm}|t|^{-\nu}. \tag{A.4}$$

One can also define amplitudes along the critical isotherm, for example

$$\chi = C^c |h|^{-\gamma/\beta\delta} \tag{A.5}$$

$$\xi = f^c |h|^{-\nu/\beta\delta} \tag{A.6}$$

$$\xi_{gap} = f_{gap}^c |h|^{-\nu/\beta\delta}. \tag{A.7}$$

Appendix B. Finite-size scaling of the free energy

Before starting the discussion on the FSS in the Ising model, let us stress that the analysis reported in this appendix is only a straightforward application of results which are already well known in the literature. They can be found for instance in [39] and are based on the results of [40] and on the exact solution of the Ising model on a finite lattice obtained by Kaufman [12]. We decided to write it down in this appendix all the same so as to make this paper as self-contained as possible. We shall use the same notations as [39] to simplify the comparison. This means in particular that we shall use the letter R to denote the finite size of the lattice in the transverse direction, which is denoted in the rest of the paper by L_1.

The aim of this appendix is to obtain the explicit form of the FSS of the free energy of the Ising model on a rectangular lattice of size $R \times \infty$, which is exactly the geometry we considered in our TM work. To obtain this result we shall work in the framework of the S-matrix approach to two-dimensional integrable models and shall in particular use the so-called thermodynamic Bethe ansatz (TBA). In the TBA one looks at the theory defined on an infinitely long cylinder with circumference R. The only free parameter of the theory is $r \equiv mR$ where m is the lowest mass of the model. The goal is to extract the behaviour of the free energy $E_0(R)$ and of the lowest mass $m(R)$ (and possibly, in models more complicated than the Ising one, of higher massive states) as a function of R. The geometry of the TBA is exactly the one which we have in our TM setting. The parameter r is the ratio L/ξ in our notations and the only thing that we must require to compare our findings with the TBA analysis is that we should be in the 'scaling region', i.e. that $\xi \gg 1$ so that we may neglect the lattice artefacts.

The TBA results depend on the entries of the S-matrix. They are in general very complicated. Thus it is usually impossible to find the exact FSS functions for any value of r

(one has usually to resort to perturbative expansions for small or large values of r). However, the thermal perturbation of the Ising model is so simple (the S-matrix is simply $S = -1$ as a consequence of the fact that the model can be mapped to a free massive field theory) that in this case the explicit expression for any value of r can be obtained.

Appendix B.1. Free energy at $h = 0$

The TBA prediction for the FSS behaviour of the free energy, in the HT phase of the two-dimensional Ising model at $h = 0$, is[7]

$$F(R) = F(\infty) - \frac{\pi \, c(r)}{6R^2} \tag{B.1}$$

where, following the standard TBA machinery (see p 668 of [39]), $c(r)$ is given by

$$c(r) = \frac{6}{\pi^2} \int_0^\infty d\theta \, r \cosh(\theta) \ln(1 + e^{-r \cosh \theta}). \tag{B.2}$$

The integral can be exactly evaluated and gives

$$c(r) = \frac{6r}{\pi^2} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} K_1(kr) \tag{B.3}$$

where K_1 is the modified Bessel function of the second kind.

The normalization of equation (B.1) could seem strange, but it is chosen such that in the $r \to 0$ limit (i.e. at the critical point) the function $c(r)$ exactly becomes the central charge of the Ising model $c = 1/2$. The idea behind this choice is the so-called c-theorem of Zamolodchikov, which states that the FSS of any two-dimensional integrable model can be parametrized by equation (B.1), where $c(r)$ is a 'running central charge', which interpolates in general between two critical points and, in this case, between the central charge of the Ising model and the value $c(\infty) = 0$, which is the appropriate one for a massive theory.

The limit in which we are interested is $r \gg 1$. In this limit, we can use the asymptotic expansion for the Bessel functions and equations (B.1), (B.3) become

$$F(R) = F(\infty) - \sqrt{\frac{m}{2\pi R^3}} \sum_{k=1}^\infty e^{-mkR} \frac{(-1)^{k-1}}{k\sqrt{k}} (1 + \cdots) \tag{B.4}$$

where the dots stand for the o(1) terms.

Appendix B.2. Derivatives of the free energy at $h = 0$

Since we are interested in the derivatives with respect to h of the free energy, we must try to extend equation (B.4) in the $h \neq 0$ plane. It is quite reasonable to assume that in (B.4) the h dependence can only be hidden in the mass term. Moreover, we know from [41, 42] that this dependence (for small values of h) is well described by an analytic even function of h, i.e.

$$m(h) = m(1 + a \, h^2 + b \, h^4 + \cdots) \tag{B.5}$$

with a and b unknown constants.

Inserting this expression in equation (B.4) and performing the derivatives we find

$$\chi_{2n}(R) - \chi_{2n}(\infty) = R^{2n-3/2}(g_{2n,1}(1/R)e^{-mR} + g_{2n,2}(1/R)e^{-2mR} + g_{2n,3}(1/R)e^{-3mR} + \cdots) \tag{B.6}$$

where the $g_{2n,i}(1/R)$ may be expanded in positive powers of $1/R$ (whose coefficients could be in principle computed as functions of a, b and m). This is the expression quoted in the text as equation (3.13).

---

[7] Notice that in [39] the authors study, instead of the free energy $F(R)$, the quantity $E_0(R)$ that is related to $F(R)$ by $E_0(R) = R[F(R) - F(\infty)]$.

Table 12. Results of the IA (3.14) applied to the test function $E(\xi = 4, R)$, cf equation (B.7).

R	$E^{(0)}$	$E^{(1)}$	$E^{(2)}$	$E^{(3)}$	$E^{(4)}$
12	1.002 919 604				
13	1.003 825 6116				
14	1.004 110 3404	1.004 240 8301			
15	1.004 027 2017	1.004 045 9912			
16	1.003 741 3786	1.004 144 4428	1.004 111 3946		
17	1.003 358 1849	1.004 866 2118	1.004 030 4410		
18	1.002 942 8048	1.008 303 4836	1.003 952 5998	1.002 005 7776	
19	1.002 533 8931	0.976 683 1739	1.005 203 2241	1.004 025 8799	
20	1.002 152 8510	0.996 943 1584	0.989 031 3495	1.004 042 3723	1.004 042 5080
21	1.001 810 0907	0.998 741 1366	0.998 916 2382	0.995 166 3087	1.004 025 9105
22	1.001 509 2173	0.999 348 0457	0.999 657 2961	0.999 717 3548	0.998 174 8024
23	1.001 249 7886	0.999 625 8634	0.999 860 3960	0.999 937 0740	0.999 948 2199
24	1.001 029 1081	0.999 772 2737	0.999 935 3998	0.999 979 3168	0.999 989 3714
25	1.000 843 3668	0.999 855 9476	0.999 967 5467	0.999 991 6598	0.999 996 7552
26	1.000 688 3533	0.999 906 3496	0.999 982 7009	0.999 996 2157	0.999 998 8812
27	1.000 559 8763	0.999 937 8507	0.999 990 3522	0.999 998 1547	0.999 999 5914
28	1.000 454 0024	0.999 958 0850	0.999 994 4245	0.999 999 0581	0.999 999 8462

Appendix B.3. A test of the iterative algorithm for the infinite-volume extrapolation

In order to further check the IA (3.14) for the infinite-size extrapolation, we apply it to a test function of the type (B.6). We consider the function

$$E(\xi, R) = 1 + \frac{\pi}{6R^2} \left. \frac{\partial^4 c(r)}{\partial h^4} \right|_{h=0} \tag{B.7}$$

where $c(r)$ is the function defined in equation (B.3) and

$$r = \frac{R}{\xi} \left(1 + h^2 + h^4\right). \tag{B.8}$$

Clearly, for $R \to \infty$, $E(\xi, \infty) = 1$. We compute $E(\xi, R)$ for finite values of R in the typical range of our TM calculations, i.e. $3 \lesssim R/\xi \lesssim 7$, and apply the IA (3.14) to determine the $E(\xi, \infty)$. The comparison with the exact value gives an idea of the effectiveness of the method. Table 12 shows the results of the IA (3.14) for $\xi = 4$, analogously to table 3 for F_4. We report the results for various iteration levels (up to four); the level zero corresponds to the original data. In order to obtain from table 12 an estimate $E(\xi, \infty)$, we follow the same strategy used in section 3.1.2, i.e. we consider the largest value of R and four iterations of the IA. The error is estimated from the residual dependence on R of the results obtained with four iterations of the algorithm. We obtain $E(\xi = 4, \infty) = 1.000 000(1)$, which is in perfect agreement with the exact number. Analogous results are obtained for other values of ξ. This confirms the effectiveness of the procedure we used to perform the extrapolation to the thermodynamic limit, and the reliability of the uncertainty we considered.

Appendix C. Determination of e_h

In this appendix we present the determination of e_h. We will obtain it by analysing the scaling corrections to the free energy in the presence of a magnetic field h on the critical isotherm $t = 0$. The scaling corrections for $h \to 0$ have been determined in [4]. Using the notation of [4], we have, cf equation (84) of [4],

$$F(0, h) = f_b + A_f^l |h|^{\frac{16}{15}} (1 + A_{f,b}^l |h|^{\frac{14}{15}} + A_{f,1}^l |h|^{\frac{16}{15}} + A_{f,2}^l |h|^{\frac{22}{15}} + A_{f,3}^l |h|^{\frac{30}{15}} + \cdots). \tag{C.1}$$

Each of these amplitudes has a precise physical meaning. Let us look at them in detail:

f_b denotes the bulk contribution to the free energy and can be obtained from the exact solution of the Ising model on the lattice at the critical point. Explicitly $f_b = \frac{2}{\pi}G + \frac{1}{2}\log 2$, where G is Catalan's constant.

A_f^l is the amplitude of the singular part of the free energy. It can be evaluated exactly in the framework of the S-matrix approach to the model. Its value is $A_f^l = 0.992\,7995\ldots$ [2,4].

$A_{f,b}^l$ is the first correction (proportional to h^2) due to the bulk part of the free energy. It can be related to the constant term $D_0 = -0.104\,133\,245\ldots$ [43] appearing in the expansion in powers of t of the susceptibility at $h = 0$. Its value is $A_{f,b}^l = D_0/(2A_f^l) = -0.052\,4442\ldots$

$A_{f,1}^l$ is due to the $T\bar{T}$, T^2 and $\bar{T}^2$ irrelevant operators in the Hamiltonian. This amplitude turns out to be compatible with zero [4].

$A_{f,2}^l$ is due to the $b_t h^2$ term in u_t.

$A_{f,3}^l$ is due to the $e_h h^2$ term in u_h.

In principle one could use the TM data to estimate all these constants. In practice this procedure works only for the first unknown amplitude. All the higher ones are then 'shadowed' by the first.

The problem with e_h is that it appears at a rather high level. In [4] we were able to fix exactly the amplitudes only up to $A_{f,1}^l$. The first unknown one was $A_{f,2}^l$, which was then estimated numerically, obtaining

$$0.020 < A_{f,2}^l < 0.022. \tag{C.2}$$

It was impossible to give any reliable estimate for $A_{f,3}^l$. The main progress of the present paper with respect to that analysis is that, thanks to the exact calculation of b_t performed in [5,14], we are now in a position to estimate exactly also $A_{f,2}^l$. A direct calculation gives

$$A_{f,2}^l = \frac{A_E^l \pi E_0}{A_f^l 8\beta_c} \tag{C.3}$$

where A_f^l was defined above, E_0 is given by equation (3.21) and A_E^l is defined by the singular behaviour of the internal energy, i.e. $E_{int}(t = 0, h) = \frac{1}{2}\partial F/\partial\beta = E_{bulk} + A_E^l h^{8/15} + \cdots$ Numerically, $A_E^l = 0.580\,51\ldots$ [4,44]. Substituting in equation (C.3), we find

$$A_{f,2}^l = 0.021\,0115\ldots \tag{C.4}$$

in perfect agreement with the estimate of [4].

Substituting in equation (C.1), we may now estimate numerically the amplitude $A_{f,3}^l$, which is related to e_h by

$$A_{f,3}^l = \tfrac{16}{15}e_h. \tag{C.5}$$

A standard application of the fitting procedure discussed in [4], using as input data those reported in table 10 of [4], gives

$$e_h = -0.007\,27(15). \tag{C.6}$$

References

[1] Onsager L 1944 *Phys. Rev.* **65** 117
[2] Zamolodchikov A B 1989 *Adv. Stud. Pure Math.* **19** 641
    Zamolodchikov A B 1988 *Int. J. Mod. Phys.* A **3** 743
[3] McCoy B M and Wu T T 1973 *The Two Dimensional Ising Model* (Cambridge: Harvard University Press)

McCoy B M 1995 *Statistical Mechanics and Field Theory* ed V V Bazhanov and C J Burden (Singapore: World Scientific)

[4] Caselle M and Hasenbusch M 2000 *Nucl. Phys.* B **579** 667
[5] Caselle M, Hasenbusch M, Pelissetto A and Vicari E 2000 *J. Phys. A: Math. Gen.* **33** 8171
[6] Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* 3rd edn (Oxford: Clarendon)
[7] Guida R and Zinn-Justin J 1997 *Nucl. Phys.* B **489** 626
[8] Schofield P 1969 *Phys. Rev. Lett.* **22** 606
[9] Schofield P, Lister J D and Ho J T 1969 *Phys. Rev. Lett.* **23** 1098
[10] Josephson B D 1969 *J. Phys. C: Solid State Phys.* **2** 1113
[11] Campostrini M, Pelissetto A, Rossi P and Vicari E 1999 *Phys. Rev.* E **60** 3526
[12] Kaufman B 1949 *Phys. Rev.* **76** 1232
[13] Gartenhaus S and McCullough W S 1988 *Phys. Rev.* B **38** 11688
[14] Nickel B 1999 *J. Phys. A: Math. Gen.* **32** 3889
     Nickel B 2000 *J. Phys. A: Math. Gen.* **33** 1693
[15] Orrick W P, Nickel B, Guttmann A J and Perk J H H 2000 *J. Stat. Phys.* **102** 795
     Orrick W P, Nickel B, Guttmann A J and Perk J H H 2000 Critical behaviour of the two-dimensional Ising susceptibility *Preprint* cond.mat/0009059
[16] Calabrese P, Caselle M, Celi A, Pelissetto A and Vicari E 2000 *J. Phys. A: Math. Gen.* **33** 8155
[17] Aharony A and Fisher M E 1983 *Phys. Rev.* B **27** 4394
[18] Salas J and Sokal A D 2000 *J. Stat. Phys.* **98** 551
     Salas J and Sokal A D 1999 Universal amplitude ratios in the critical two-dimensional Ising model on a torus *Preprint* cond.mat/9904038v1
[19] Wegner F J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M Green (New York: Academic) p 7
[20] Kong X P 1986 (unpublished) cited in [13]
[21] Balog J, Niedermaier M, Niedermayer F, Patrascioiu A, Seiler E and Weisz P 2000 *Nucl. Phys.* B **583** 614
[22] Pelissetto A and Vicari E 1998 *Nucl. Phys.* B **519** 626
[23] Butera P and Comi M 1996 *Phys. Rev.* B **54** 15828
[24] Zinn S, Lai S-N and Fisher M E 1996 *Phys. Rev.* E **54** 1176
[25] Kim J-K 2000 *J. Phys. A: Math. Gen.* **33** 2675
     (Kim J-K 1999 *Preprint* cond.mat/9905138)
[26] Pelissetto A and Vicari E 2000 *Nucl. Phys.* B **575** 579
[27] Le Guillou J C and Zinn-Justin J 1977 *Phys. Rev. Lett.* **39** 95
     Le Guillou J C and Zinn-Justin J 1980 *Phys. Rev.* B **21** 3976
[28] Orlov E V and Sokolov A I 2000 *Fiz. Tverd. Tela* **42** 2087 (Engl. transl. *Phys. Solid. State* **42** 2151)
     (a shorter English version appears as *Preprint* hep-th/0003140)
[29] Bender C M and Boettcher S 1992 *Phys. Rev.* D **48** 4919
     Bender C M and Boettcher S 1995 *Phys. Rev.* D **51** 1875
[30] Pelissetto A and Vicari E 1998 *Nucl. Phys.* B **522** 605
[31] Sokolov A I and Orlov E V 1998 *Phys. Rev.* B **58** 2395
[32] Griffiths R B 1967 *Phys. Rev.* **158** 176
[33] Wu T T, McCoy B M, Tracy C A and Barouch E 1976 *Phys. Rev.* B **13** 316
[34] Delfino G 1998 *Phys. Lett.* B **419** 291
[35] Campostrini M, Pelissetto A, Rossi P and Vicari E 1996 *Phys. Rev.* B **54** 7301
     Campostrini M, Pelissetto A, Rossi P and Vicari E 1998 *Phys. Rev.* E **57** 184
[36] Pelissetto A and Vicari E 1999 *Nucl. Phys.* B **540** 639
[37] Campostrini M, Pelissetto A, Rossi P and Vicari E 2000 *Phys. Rev.* B **62** 5843
[38] Campostrini M, Hasenbusch M, Pelissetto A, Rossi P and Vicari E 2000 Critical behavior of the *XY* universality class *Preprint* cond-mat/0010360 (*Phys. Rev.* B at press)
[39] Klassen T R and Melzer E 1991 *Nucl. Phys.* B **350** 635
[40] Ferdinand A E and Fisher M E 1969 *Phys. Rev.* **185** 832
[41] Delfino G, Mussardo G and Simonetti P 1996 *Nucl. Phys.* B **473** 469
[42] McCoy B M and Wu T T 1978 *Phys. Rev.* D **18** 1259
[43] Kong X P, Au-Yang H and Perk J H H 1986 *Phys. Lett.* A **116** 54
[44] Fateev V, Lukyanov S, Zamolodchikov A and Zamolodchikov Al 1998 *Nucl. Phys.* B **516** 652